

LOCALLY UNMIXED MODULES AND LINEARLY EQUIVALENT IDEAL TOPOLOGIES

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ABSTRACT. Let R be a commutative Noetherian ring, and let N be a non-zero finitely generated R -module. The purpose of this paper is to show that N is locally unmixed if and only if, for any N -proper ideal I of R generated by $\text{ht}_N I$ elements, the topology defined by $(IN)^{(n)}$, $n \geq 0$, is linearly equivalent to the I -adic topology.

1. INTRODUCTION

Let R denote a commutative Noetherian ring, I an ideal of R and N a non-zero finitely generated R -module. We denote by $R[It]$ (resp. $R[It, u]$) the *graded ordinary* (resp. *extended*) *Rees ring* $\bigoplus_{n \in \mathbb{N}_0} I^n t^n$ (resp. $\bigoplus_{n \in \mathbb{Z}} I^n t^n$) of R with respect to I , where t is an indeterminate and $u = t^{-1}$. Also, the *graded ordinary Rees module* $\bigoplus_{n \in \mathbb{N}_0} I^n N$ over $R[It]$ (resp. *graded extended Rees module* $\bigoplus_{n \in \mathbb{Z}} I^n N$ over $R[It, u]$) is denoted by $N[It]$ (resp. $N[It, u]$), which is finitely generated. For any multiplicatively closed subset S of R , the n th (S) -symbolic power of I with respect to N , denoted by $S(I^n N)$, is defined to be the union of $I^n N :_N s$ where s varies in S . The I -adic filtration $\{I^n N\}_{n \geq 0}$ and the (S) -symbolic filtration $\{S(I^n N)\}_{n \geq 0}$ induce topologies on N which are called the *I -adic topology* and the *(S) -symbolic topology*, respectively. These two topologies are said to be *linearly equivalent* if, there is an integer $k \geq 0$ such that $S(I^{n+k} N) \subseteq I^n N$ for all integers n . In particular, if $S = R \setminus \bigcup \{\mathfrak{p} \in \text{mAss}_R N/IN\}$, where $\text{mAss}_R N/IN$ denotes the set of minimal prime ideals of $\text{Ass}_R N/IN$, the n th (S) -symbolic power of I with respect to N , is denoted by $(IN)^{(n)}$, and the topology defined by the filtration $\{(IN)^{(n)}\}_{n \geq 0}$ is called the *symbolic topology*. The purpose of this paper is to show that N is locally unmixed if and only if, for each N -proper ideal I that is generated by $\text{ht}_N I$ elements, the I -adic and the symbolic topologies are linearly equivalent.

P. Schenzel has characterized unmixed local rings [19, Theorem 7] in terms of comparison of the topologies defined by certain filtrations. Also, D. Katz [9, Theorem 3.5] and J. Verma [21, Theorem 5.2] have proved a characterization of locally unmixed rings in terms of s -ideals. Equivalence of I -adic topology and (S) -symbolic topology has been studied, in the case $N = R$, in [9, 15, 19, 18, 17], and has led to some interesting results.

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Let $\mathfrak{p} \in \text{Supp}(N)$. Then N -height of \mathfrak{p} , denoted by $\text{ht}_N \mathfrak{p}$, is defined to be the supremum of lengths of chains of prime ideals of $\text{Supp}(N)$ terminating with \mathfrak{p} . We have $\text{ht}_N \mathfrak{p} = \dim_{R_{\mathfrak{p}}} N_{\mathfrak{p}}$. We shall say an ideal I of R is N -proper if $N/IN \neq 0$, and, when this is the case, we define the N -height of I (written $\text{ht}_N I$) to be

$$\inf\{\text{ht}_N \mathfrak{p} : \mathfrak{p} \in \text{Supp}(N) \cap V(I)\} \\ (= \inf\{\text{ht}_N \mathfrak{p} : \mathfrak{p} \in \text{Ass}_R(N/IN)\}).$$

If (R, \mathfrak{m}) is local, then \widehat{R} (resp. \widehat{N}) denotes the completion of R (resp. N) with respect to the \mathfrak{m} -adic topology. In particular, for any $\mathfrak{p} \in \text{Spec}(R)$, we denote $\widehat{R}_{\mathfrak{p}}$ and $\widehat{N}_{\mathfrak{p}}$ the $\mathfrak{p}R_{\mathfrak{p}}$ -adic completion of $R_{\mathfrak{p}}$ and $N_{\mathfrak{p}}$, respectively. Then N is said to be an *unmixed module* if for any $\mathfrak{p} \in \text{Ass}_{\widehat{R}} \widehat{N}$, $\dim \widehat{R}/\mathfrak{p} = \dim N$. More generally, if R is not necessarily local and N is non-zero finitely generated, N is a *locally unmixed module* if for any $\mathfrak{p} \in \text{Supp}(N)$, $N_{\mathfrak{p}}$ is an unmixed $R_{\mathfrak{p}}$ -module.

As the main result of this paper we characterize the locally unmixed property of a non-zero finitely generated R -module N in terms of the linearly equivalence of the topologies defined by $\{I^n N\}_{n \geq 0}$ and $\{(IN)^{(n)}\}_{n \geq 0}$, for certain N -proper ideals I of R . More precisely we shall show that:

Theorem 1.1. *Let R be a Noetherian ring and N a non-zero finitely generated R -module. Then the following conditions are equivalent:*

- (i) *N is locally unmixed.*
- (ii) *For each N -proper ideal I of R that is generated by $\text{ht}_N I$ elements, the topology given by $\{(IN)^{(n)}\}_{n \geq 0}$ is linearly equivalent to the I -adic topology on N .*

One of our tools for proving Theorem 1.1 is the following, which plays a key role in this paper. Recall that a prime ideal \mathfrak{p} of R is called a *quitesential prime ideal* of I with respect to N precisely when there exists $\mathfrak{q} \in \text{Ass}_{\widehat{R}_{\mathfrak{p}}} \widehat{N}_{\mathfrak{p}}$ such that $\text{Rad}(I\widehat{R}_{\mathfrak{p}} + \mathfrak{q}) = \mathfrak{p}\widehat{R}_{\mathfrak{p}}$. The set of quitesential primes of I is denoted by $Q(I, N)$. Then, the set of *es-sential primes* of I with respect to N , denoted by $E(I, N)$, is defined to be the set $\{\mathfrak{q} \cap R \mid \mathfrak{q} \in Q(uR[It, u], N[It, u])\}$.

Theorem 1.2. *Let R denote a Noetherian ring, N a non-zero finitely generated R -module and I a N -proper ideal of R such that $E(I, N) = \text{mAss}_R N/IN$. Then, the I -adic topology $\{I^n N\}_{n \geq 0}$ and the topology defined by $\{(I^n N)^{(n)}\}_{n \geq 0}$ are linearly equivalent.*

The proof of Theorem 1.2 is given in 1.13.

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity, N will be a non-zero finitely generated R -module, and I will be an N -proper ideal of R , i.e., $N/IN \neq 0$. For each R -module L , we denote by $\text{mAss}_R L$ the set of minimal primes of $\text{Ass}_R L$. For any ideal J of R , the *radical of J* , denoted by $\text{Rad}(J)$, is defined to be the set $\{x \in R : x^n \in J \text{ for some } n \in \mathbb{N}\}$. For any unexplained notation and terminology we refer the reader to [6] or [12].

2. THE RESULTS

The main result of this section is to show that a non-zero finitely generated module N over a Noetherian ring R is locally unmixed if and only if, for any N -proper ideal I of R that can be generated by $\text{ht}_N I$ elements, the topologies defined by $\{I^n N\}_{n \geq 0}$ and $\{(IN)^{(n)}\}_{n \geq 0}$, on N , are linearly equivalent. We begin with the following remark.

Remark 2.1. Let R be a Noetherian ring and N a finitely generated R -module. For a submodule M of N and an ideal I of R , the increasing sequence of submodules

$$M \subseteq M :_N I \subseteq M :_N I^2 \subseteq \cdots \subseteq M :_N I^n \subseteq \cdots$$

becomes stationary. Denote its ultimate constant value by $M :_N \langle I \rangle$. Note that $M :_N \langle I \rangle = M :_N I^n$ for all large n . Let

$$M = Q_1 \cap \cdots \cap Q_r \cap Q_{r+1} \cap \cdots \cap Q_s$$

be an irredundant primary decomposition of M , with $I \subseteq \text{Rad}(Q_i :_R N)$, exclusively for $r+1 \leq i \leq s$. Then, from the definition, it easily follows that $M :_N \langle I \rangle = Q_1 \cap \cdots \cap Q_r$. Therefore

$$\text{Ass}_R N / (M :_N \langle I \rangle) = \{\mathfrak{p} \in \text{Ass}_R N / M : I \not\subseteq \mathfrak{p}\} = \text{Ass}_R(N/M) \setminus V(I).$$

Now we can state and prove the following lemma. Here $D_I(L)$ denotes the ideal transform of the R -module L with respect to an ideal I of R (see [5, 2.2.1]).

Lemma 2.2. *Let (R, \mathfrak{m}) be local (Noetherian) ring, I an ideal of R and N a non-zero finitely generated R -module such that $\text{depth } N > 0$. Then, for all integers $n \geq 0$, we have*

$$I^n N :_N \langle \mathfrak{m} \rangle \subseteq D_{\mathfrak{m}}(I^n N).$$

Proof. The assertion follows from [5, Corollary 2.2.18] and the fact that $\text{depth } I^n N > 0$ for all integers $n \geq 0$. \square

The next result concerns the associated prime ideals of the Rees module $N[It]$ for a non-zero finitely generated module N over a Noetherian ring R and an ideal I in R .

Proposition 2.3. *Let R be a Noetherian ring, I an ideal of R and N a non-zero finitely generated R -module. Then*

$$\text{Ass}_{R[It]} N[It] = \{\oplus_{n \geq 0} (I^n \cap \mathfrak{p}) : \mathfrak{p} \in \text{Ass}_R N\}.$$

Proof. Let $\mathfrak{q} \in \text{Ass}_{R[It]} N[It]$. Then in view of [6, Lemma 1.5.6] there exists a homogenous element x of $N[It]$ such that $\mathfrak{q} = \text{Ann}_{R[It]} x$. Suppose that $x \in I^v N$ for some integer $v \geq 0$. Then we have

$$\mathfrak{q} = (0 :_{R[It]} x) = \oplus_{n \geq 0} (0 :_R x) \cap I^n.$$

Now, it is easy to see that $\mathfrak{p} := (0 :_R x)$ is a prime ideal of R and so $\mathfrak{p} \in \text{Ass}_R N$. Hence $\mathfrak{q} = \oplus_{n \geq 0} (I^n \cap \mathfrak{p})$ for some $\mathfrak{p} \in \text{Ass}_R N$. Conversely, let $\mathfrak{p} \in \text{Ass}_R N$ and $\mathfrak{p} = (0 :_R x)$ for an element $x \in N$. Then

$$\mathfrak{q} := (0 :_{R[It]} x) = \oplus_{n \geq 0} (I^n \cap \mathfrak{p})$$

is a prime ideal of $\text{Ass}_{R[It]} N[It]$, because $R[It]/\mathfrak{q} \cong R/\mathfrak{p}[(I + \mathfrak{p}/\mathfrak{p})t]$ is a domain. \square

Definition 2.4. Let R be a Noetherian ring and N an R -module. A decreasing sequence $\{N_n\}_{n \geq 0}$ of submodules of N is called a *filtration* of N . If I is an ideal of R , then the filtration $\{N_n\}_{n \geq 0}$ is called *I -filtration* whenever $IN_n \subseteq N_{n+1}$ for all integers $n \geq 0$.

Lemma 2.5. Let R be a Noetherian ring, I an ideal of R and N an R -module. Let $\{N_n\}_{n \geq 0}$ be an I -filtration of submodules of N such that the ordinary Rees module $N[It]$ is finitely generated over $R[It]$. Then there exists an integer k such that $N_{n+k} = I^n N_k$, for all integers $n \geq 0$.

Proof. The result follows easily from [7, Lemma 2.5.4]. \square

Corollary 2.6. Let (R, \mathfrak{m}) be a local (Noetherian) ring and I an ideal of R . Let N be an R -module and set $N_n = I^n N :_N \langle \mathfrak{m} \rangle$ for each integer $n \geq 0$. Suppose that the module $\bigoplus_{n \geq 0} N_n$ is finitely generated over the ordinary Rees ring $R[It]$. Then there is an integer k such that $I^{n+k} N :_N \langle \mathfrak{m} \rangle \subseteq I^n N$, for all integer $n \geq 0$.

Proof. As $I(I^n N :_N \langle \mathfrak{m} \rangle) \subseteq I^{n+1} N :_N \langle \mathfrak{m} \rangle$, for all integers $n \geq 0$, the claim follows from Lemma 2.5. \square

Definition 2.7. Let (R, \mathfrak{m}) be a local (Noetherian) ring, I an ideal of R and N an R -module. We define the R -module $D(I, N)$ as the following:

$$D(I, N) := \bigoplus_{n \geq 0} D_{\mathfrak{m}}(I^n N).$$

As $D_{\mathfrak{m}}(\cdot)$ is an R -linear and left exact functor, it follows that $\{D_{\mathfrak{m}}(I^n N)\}_{n \geq 0}$ is a decreasing sequence and $ID_{\mathfrak{m}}(I^n N) \subseteq D_{\mathfrak{m}}(I^{n+1} N)$ for all integers $n \geq 0$. Hence $D(I, N)$ is an $R[It]$ -module, by Lemma 2.5.

Lemma 2.8. Let R be a Noetherian ring, I an ideal of R and N a finitely generated R -module. Then the following conditions are equivalent:

- (i) $D_I(N)$ is a finitely generated R -module.
- (ii) For all $\mathfrak{p} \in \text{Ass}_R N$, the R/\mathfrak{p} -module $D_{I(R/\mathfrak{p})}(R/\mathfrak{p})$ is finitely generated.

Proof. See [4, Lemma 3.3]. \square

Proposition 2.9. Let (R, \mathfrak{m}) be a local (Noetherian) ring, I an ideal of R and N a finitely generated R -module. Then the following conditions are equivalent:

- (i) $D(I, N)$ is a finitely generated $R[It]$ -module.
- (ii) For all $\mathfrak{p} \in \text{Ass}_R N$, the module $\bigoplus_{n \geq 0} D_{\mathfrak{m}}(I^n + \mathfrak{p}/\mathfrak{p})$ is finitely generated over the Rees ring $R/\mathfrak{p}[(I + \mathfrak{p}/\mathfrak{p})t]$.

Proof. In order to prove the implication (i) \implies (ii), suppose that $\mathfrak{p} \in \text{Ass}_R N$. Then in view of Proposition 2.3, there exists $\mathfrak{q} \in \text{Ass}_{R[It]}(\bigoplus_{n \geq 0} I^n N)$ such that $\mathfrak{q} = \bigoplus_{n \geq 0} (I^n \cap \mathfrak{p})$. Since

$$D(I, N) \cong D_{\mathfrak{m}}(\bigoplus_{n \geq 0} I^n N) \cong D_{\mathfrak{m}R[It]}(\bigoplus_{n \geq 0} I^n N),$$

is a finitely generated $R[It]$ -module, it follows from Lemma 2.8 that the $R[It]/\mathfrak{q}$ -module $D_{\mathfrak{m}(R[It]/\mathfrak{q})}(R[It]/\mathfrak{q})$ is finitely generated. Now, as

$$R[It]/\mathfrak{q} \cong R/\mathfrak{p}[(I + \mathfrak{p}/\mathfrak{p})t], \text{ and } D_{\mathfrak{m}(R[It]/\mathfrak{q})}(R[It]/\mathfrak{q}) \cong \bigoplus_{n \geq 0} D_{\mathfrak{m}}(I^n + \mathfrak{p}/\mathfrak{p}),$$

we deduce that the $R/\mathfrak{p}[(I + \mathfrak{p}/\mathfrak{p})t]$ -module $\bigoplus_{n \geq 0} D_{\mathfrak{m}}(I^n + \mathfrak{p}/\mathfrak{p})$ is finitely generated.

Now, we show the conclusion (ii) \implies (i). To do this end, let $\mathfrak{q} \in \text{Ass}_{R[It]} N[It]$. Then, by virtue of Proposition 2.3, there exists $\mathfrak{p} \in \text{Ass}_R N$ such that $\mathfrak{q} = \bigoplus_{n \geq 0} (I^n \cap \mathfrak{p})$. Since

$$R[It]/\mathfrak{q} \cong R/\mathfrak{p}[(I + \mathfrak{p}/\mathfrak{p})t] \text{ and } D_{\mathfrak{m}(R[It]/\mathfrak{q})}(R[It]/\mathfrak{q}) \cong \bigoplus_{n \geq 0} D_{\mathfrak{m}}(I^n + \mathfrak{p}/\mathfrak{p}),$$

it follows from Lemma 2.8 that the $R[It]$ -module $D_{\mathfrak{m}R[It]}(N[It])$ is finitely generated, and so the $R[It]$ -module $\bigoplus_{n \geq 0} D_{\mathfrak{m}}(I^n N)$ is finitely generated, as required. \square

The next proposition gives us a criterion for the finiteness of $R[It]$ -module $D_{\mathfrak{m}R[It]}(N[It])$, whenever (R, \mathfrak{m}) is a local ring and N is a finitely generated module over R . To this end, let us, firstly, recall the important notion *analytic spread of I with respect to N* , over a local ring (R, \mathfrak{m}) , introduced by Brodmann in [3]:

$$l(I, N) := \dim N[It]/(\mathfrak{m}, u)N[It],$$

in the case $N = R$, $l(I, N)$ is the classical analytic spread $l(I)$ of I , introduced by Northcott and Rees (see [14]).

Proposition 2.10. *Let (R, \mathfrak{m}) be a local (Noetherian) ring and I an ideal of R . Let N be a finitely generated R -module such that $l(I\hat{R} + \mathfrak{p}/\mathfrak{p}) < \dim \hat{R}/\mathfrak{p}$ for all $\mathfrak{p} \in \text{Ass}_{\hat{R}} \hat{N}$. Then the $R[It]$ -module $D_{\mathfrak{m}R[It]}(N[It])$ is finitely generated, and $\text{depth } N > 0$.*

Proof. It is easy to see that

$$D_{\mathfrak{m}R[It]}(N[It]) \otimes_{R[It]} \hat{R}[(I\hat{R})t] \cong D_{\mathfrak{m}\hat{R}[(I\hat{R})t]}(\bigoplus_{n \geq 0} I^n \hat{N}),$$

and so by faithfully flatness of $\hat{R}[(I\hat{R})t]$ over $R[It]$, it is enough for us to show that the $\hat{R}[(I\hat{R})t]$ -module $\bigoplus_{n \geq 0} D_{\mathfrak{m}\hat{R}[(I\hat{R})t]}(I^n \hat{N})$ is finitely generated. In order to do this, in view of Proposition 2.9, it is enough to show that $\bigoplus_{n \geq 0} D_{\mathfrak{m}}(I^n \hat{R} + \mathfrak{p}/\mathfrak{p})$ is finitely generated over $\hat{R}/\mathfrak{p}[(I\hat{R} + \mathfrak{p}/\mathfrak{p})t]$ for all $\mathfrak{p} \in \text{Ass}_{\hat{R}} \hat{N}$. But this follows easily from [19, Proposition] and the assumption $l(I\hat{R} + \mathfrak{p}/\mathfrak{p}) < \dim \hat{R}/\mathfrak{p}$. \square

Remark 2.11. Before bringing the next result we fix a notation, which is employed by P. Schenzel in [18] in the case $N = R$. Let S be a multiplicatively closed subset of a Noetherian ring R . For a submodule M of a finitely generated R -module N , we use $S(M)$ to denote the submodule $\bigcup_{s \in S} (M :_N s)$. Note that the primary decomposition of $S(M)$ consists of the intersection of all primary components of M whose associated prime ideals do not meet S . In other words

$$\text{Ass}_R N/S(M) = \{\mathfrak{p} \in \text{Ass}_R N/M : \mathfrak{p} \cap S = \emptyset\}.$$

In particular, if $S = R \setminus \bigcup \{\mathfrak{p} \in \text{mAss}_R N/IN\}$, then for any $n \in \mathbb{N}$, $S(I^n N)$ is denoted by $(IN)^{(n)}$, where I is an ideal of R .

The following lemma is needed in the proof of Theorem 2.13.

Lemma 2.12. *Let R be a Noetherian ring and N an R -module. Let M and L be two submodules of R such that $M_{\mathfrak{p}} \subseteq L_{\mathfrak{p}}$ for all $\mathfrak{p} \in \text{Ass}_R N/L$. Then $M \subseteq L$.*

Proof. The assertion follows from the fact that $\text{Ass}_R(M + L/L) \subseteq \text{Ass}_R N/L$. \square

Following, we investigate a fundamental characterization for linearly equivalence between the I -adic and symbolic topologies on a finitely generated R -module N , for certain ideal I of R . This result plays a key role in the proof of the main theorem.

To this end, recall that, in [16], L.J. Ratliff, Jr., (resp. in [2] Brodmann) introduced the interesting set of associated primes $\bar{A}^*(I) := \text{Ass}_R R/(I^n)_a$ (resp. $A^*(I, N) := \text{Ass}_R N/I^n N$), for large n . Here I_a denotes the integral closure of I in R , i.e., I_a is the ideal of R consisting of all elements $x \in R$ which satisfy an equation $x^n + r_1 x^{n-1} + \cdots + r_n = 0$, where $r_i \in I^i, i = 1, \dots, n$.

Moreover, recall that a local ring (R, \mathfrak{m}) is said to be a *quasi-unmixed ring* if for every $\mathfrak{p} \in \text{mAss } \widehat{R}$, the condition $\dim \widehat{R}/\mathfrak{p} = \dim R$ is satisfied.

Theorem 2.13. *Let R be a Noetherian ring, I an ideal of R and let N be a finitely generated R -module such that $E(I, N) = \text{mAss}_R N/IN$. Then, the I -adic topology, $\{I^n N\}_{n \geq 0}$ and the topology defined by the filtration $\{(IN)^{(n)}\}_{n \geq 0}$ are linearly equivalent.*

Proof. Let $\mathfrak{q} \in A^*(I, N) \setminus \text{mAss}_R N/IN$ and let $z \in \text{Ass}_{\widehat{R}_{\mathfrak{q}}} \widehat{N}_{\mathfrak{q}}$. Then, by assumption, $\mathfrak{q} \notin E(I, N)$. Hence, in view of [1, Lemma 3.2], $\mathfrak{q} R_{\mathfrak{q}} \notin E(IR_{\mathfrak{q}}, N_{\mathfrak{q}})$, and so it follows from [1, Proposition 3.6] that $\widehat{\mathfrak{q} R_{\mathfrak{q}}}/z \notin E(\widehat{IR}_{\mathfrak{q}} + z/z)$. Thus by virtue of [11, Lemma 2.1], $\widehat{\mathfrak{q} R_{\mathfrak{q}}}/z \notin \bar{A}^*(\widehat{IR}_{\mathfrak{q}} + z/z)$. As $\widehat{R}_{\mathfrak{q}}/z$ is quasi-unmixed, it follows from McAdam's result [10, Proposition 4.1] that

$$l(\widehat{IR}_{\mathfrak{q}} + z/z) < \dim \widehat{R}_{\mathfrak{q}}/z. \quad (\dagger)$$

Now, we show that there exists a non-negative integer k such that $(IN)^{(n+k)} \subseteq I^n N$ for all integers $n \geq 0$. To do this, it is easy to see that, $(IN)_{\mathfrak{p}}^{(s)} \subseteq (I^s N)_{\mathfrak{p}}$ for all $\mathfrak{p} \in \text{mAss}_R N/IN$ and for all integers $s \geq 0$. Moreover, if for every $\mathfrak{q} \in A^*(I, N) \setminus \text{mAss}_R N/IN$ there exists an integer $k_{\mathfrak{q}}$ such that

$$(IN)_{\mathfrak{q}}^{(n+k_{\mathfrak{q}})} \subseteq (I^n N)_{\mathfrak{q}},$$

then by considering

$$k := \max\{k_{\mathfrak{q}} : \mathfrak{q} \in A^*(I, N) \setminus \text{mAss}_R N/IN\},$$

one easily sees that $(IN)^{(n+k)} \subseteq I^n N$. Since both $A^*(I, N)$ and $\text{mAss}_R N/IN$ behave well under localization, we may assume by localizing at \mathfrak{q} that (R, \mathfrak{m}) is a local ring.

Now, we use induction on $\dim N/IN := d$. It is clear that $d \geq 1$. Now, if $d = 1$, then, as $\text{Ass}_R N/IN \subseteq \text{Supp } N/IN$ and $\mathfrak{m} \in \text{Supp } N/IN$ it follows that the only possible embedded prime of $\text{Ass}_R N/IN$ is \mathfrak{m} , and so in view of Remark 2.1 we have

$$I^s N :_N \langle \mathfrak{m} \rangle = (IN)^{(s)}$$

for all integers $s \geq 0$. Next, it follows from (\dagger) and Proposition 2.10 that the $R[It]$ -module $\bigoplus_{n \geq 0} D_{\mathfrak{m}}(I^n N)$ is finitely generated and $\text{depth } N > 0$. Hence in view of Lemma 2.2, the module $\bigoplus_{n \geq 0} (I^n N :_N \langle \mathfrak{m} \rangle)$ is finitely generated over the Rees ring $R[It]$, and so by virtue of Corollary 2.6, there exists an integer t such that $I^{n+t} N :_N \langle \mathfrak{m} \rangle \subseteq I^n N$ for all integers $n \geq 0$. Therefore $(IN)^{(n+k)} \subseteq I^n N$, and so the result holds for $d = 1$.

We therefore assume, inductively, that $d > 1$ and the result has been proved for smaller values of d . If $\mathfrak{q} \neq \mathfrak{m}$ and $\mathfrak{q} \in A^*(I, N)$, then

$$\dim N_{\mathfrak{q}}/IN_{\mathfrak{q}} = \text{ht}_{N/IN} \mathfrak{q} < \text{ht}_{N/IN} \mathfrak{m} = \dim N/IN = d.$$

Hence by induction hypothesis, there exists a non-negative integer $k_{\mathfrak{q}}$ such that

$$(IN)_{\mathfrak{q}}^{(n+k_{\mathfrak{q}})} \subseteq (I^n N)_{\mathfrak{q}},$$

for all integers $n \geq 0$. Now, in view of Remark 2.1,

$$\text{Ass}_R N/(I^n N :_N \langle \mathfrak{m} \rangle) = \text{Ass}_R N/I^n N \setminus V(\mathfrak{m}),$$

it follows that for all $\mathfrak{q} \in \text{Ass}_R N/(I^n N :_N \langle \mathfrak{m} \rangle)$, there exists a non-negative integer $k_{\mathfrak{q}}$ such that

$$(IN)_{\mathfrak{q}}^{(n+k_{\mathfrak{q}})} \subseteq (I^n N)_{\mathfrak{q}} \subseteq ((I^n N)_{\mathfrak{q}} :_{N_{\mathfrak{q}}} \langle \mathfrak{m} R_{\mathfrak{q}} \rangle),$$

for all integers $n \geq 0$. Hence by considering

$$k := \max\{k_{\mathfrak{q}} : \mathfrak{q} \in \text{Ass}_R N/(I^n N :_N \langle \mathfrak{m} \rangle)\},$$

we get

$$(IN)_{\mathfrak{q}}^{(n+k)} \subseteq (I^n N)_{\mathfrak{q}} :_{N_{\mathfrak{q}}} \langle \mathfrak{m} R_{\mathfrak{q}} \rangle,$$

for all $\mathfrak{q} \in \text{Ass}_R N/(I^n N :_N \langle \mathfrak{m} \rangle)$ and all integers $n \geq 0$. Therefore, by virtue of the Lemma 2.12, we have

$$(IN)^{(n+k)} \subseteq (I^n N) :_N \langle \mathfrak{m} \rangle.$$

On the other hand, in view of Corollary 2.6, there exists an integer $s \geq 0$ such that

$$I^{n+s} N :_N \langle \mathfrak{m} \rangle \subseteq I^n N$$

for all integers $n \geq 0$. Consequently

$$(IN)^{(n+k+s)} \subseteq I^{n+s} N :_N \langle \mathfrak{m} \rangle \subseteq I^n N,$$

for all integers $n \geq 0$, and thus the topologies defined by the filtrations $\{I^n N\}_{n \geq 0}$ and $\{(IN)^{(n)}\}_{n \geq 0}$ are linearly equivalent. \square

We are now ready to state and prove the main theorem of this paper, which is a new characterization of locally unmixed modules in terms of comparison of the topologies defined by certain decreasing families of submodules of finitely generated modules over a commutative Noetherian ring. One of the implications in the proof of this theorem follows from [13, Theorem 3.2].

Theorem 2.14. *Let R be a Noetherian ring and N a non-zero finitely generated R -module. Then the following conditions are equivalent:*

- (i) *N is locally unmixed.*
- (ii) *For any N -proper ideal I of R generated by $\text{ht}_N I$ elements, the I -adic topology is linearly equivalent to the symbolic topology.*

Proof. The implication (ii) \implies (i) follows easily from [13, Theorem 3.2]. In order to prove the conclusion (i) \implies (ii), let I be an N -proper ideal of R which is generated by $\text{ht}_N I$ elements. Then, in view of Theorem 2.13 it is enough for us to show that $E(I, N) = \text{mAss}_R N/IN$. Suppose that $\mathfrak{p} \in E(I, N)$, and we show that $\mathfrak{p} \in \text{mAss}_R N/IN$. Let $\text{ht}_N I := n$. Then by [13, Theorem 2.1], there exist the elements x_1, \dots, x_n in I such that $\text{ht}_N(x_1, \dots, x_i) = i$ for all $1 \leq i \leq n$. As, in view of [13, Corollary 3.11], x_1, \dots, x_n is an essential sequence on N , and the fact that $\text{egrade}(I, N) \leq \text{ht}_N I$, it follows that $\text{egrade}(I, N) = n$. Now, analogous to the proof of [8, Theorem 125], it is easy to see that I can be generated by an essential sequence of length n . Therefore by [13, Lemma 3.8], we have $\mathfrak{p} \in \text{mAss}_R N/IN$, and so $E(I, N) \subseteq \text{mAss}_R N/IN$. As the opposite inclusion is obvious, the result follows. \square

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